ON THE SOLVABILITY OF THE GENERAL PROBLEM FOR AN ANISOTROPIC LAMINAR SHELL WITHIN THE FRAMEWORK OF MEDIUM DEFLECTION THEORY

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The solvability of the nonlinear problem for an elastic shallow shell was investigated in [1] (*). The equations of nonshallow shells have certain special features and therefore require mathematical justification. The solvability of the general problem of a closed circular cylindrical shell in nonlinear formulation is the subject of [2]. In the present paper the procedure of [2] is extended to the case of an arbitrarily fastened anisotropic laminar shell of arbitrary configuration acted on by a general load and temperature field. The method consists in extending the coercivity inequality obtained for a regular piece of the shell to a piecewise-regular shell. The possibility of this extension implies the applicability of the Leray-Schauder principle, the existence of a generalized solution, the applicability of projective methods, etc. The conclusions remain valid even if a different (linear, nonlinear) calculation theory is applied over each piece of the shell.

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1. Notation. Basic relations. Let the coordinate surface σ of the shell be defined by the equation (we use the notation of [3]) and let the following conditions be fulfilled:

1) the above equation defines a one-to-one mapping of the surface σ onto some bounded domain G in the plane α_1, α_2 with the boundary Γ;

2) α_1, α_2 is an orthogonal curvilinear coordinate system and $0 < m_1 \leqslant A_i \leqslant m_2$ $(i = 1, 2)$;

3) G consists of a finite number of starlike domains [4]; Γ consists of a finite number of closed contours;

4) $\sigma \in W_p^{(3)}$ $(p > 1)$, i.e. the function $\mathbf{r}(\alpha_1, \alpha_2)$ has in G all generalized derivatives of up to third order, inclusively, summable in some power p;

5) $\Gamma \subset Л_1(m, 0)$, i.e. to a Liapunov class (**).

Medium deflection is characterized by the fact that the elongations, shear strains, and squares of the angles of rotation are negligible compared with unity [3, 5]. Under these assumptions the general formulas of [3] yield the following deformations relations:

$$2\varepsilon_{ik} = e_{ik} + e_{ki} + e_{i3}e_{k3}, \quad e_{ik} = A_i^{-1}u_{k,i} + (-1)^{i+k}(A_1A_2)^{-1}A_{i,s}u_s + k_{ik}w$$

$$e_{i3} = A_i^{-1}w_{,i} - k_{ii}u_i - k_{is}u_s$$

$$\varkappa_{ik} = -A_i^{-1}e_{k3,i} - (-1)^{i+k}(A_1A_2)^{-1}A_{i,s}e_{s3} + k_{ir}e_{kr} + \qquad (1.1)$$

*) See also I. I. Vorovich's doctoral thesis.

**) Editorial Note. Cyrillic symbol $Л$ is obviously derived from Liapunov (Ляпунов).

$$+ k_{ik}(e_{rr} + 2e_{kk}) + k_{ik}e^2_{k3} + k_{ir}\,e_{k3}e_{r3}$$

$$s = 3 - i, \quad r = 3 - k, \quad (\cdot)_{,i} = \frac{\partial}{\partial \alpha_i}(\cdot) \qquad (i, k = 1, 2)$$

As we know, the theory of shells does not provide us with a general definition of bending strain components in the general case. This means that other expressions for $\varkappa_{ik}$ are possible within the error bracket admissible in shell theory. Thus, the author of [5] takes

$$2\varkappa_{12} = \sum_{i=1}^{2} [-A_i^{-1} e_{s3,i} + (A_1 A_2)^{-1} A_{i,2} e_{i3} + k_{ii} e_{si} + k_{ii} e_{i3} e_{s3} + k_{is} e_{s3}^2] \qquad (1.2)$$

instead of (1.1).

Some remarks should be made concerning notation. As a rule, we will denote a six- or three-component vector and its components by the same letter. A three-component vector will be defined by its projections on the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{m}$ or, if it is accompanied by the degree symbol, by its projections on the orthogonal unit vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$, where $\mathbf{n}_3 = \mathbf{m}$, $\mathbf{n}_1$ is the exterior normal to Γ, and $\mathbf{n}_2$ is the vector tangent to Γ. The domain of integration will be indicated in the differential appearing in the integrand.

We shall also use the following abbreviations:

$$\boldsymbol{\eta} = (e_{13}, e_{23}, -{}^1/_2), \qquad \boldsymbol{\xi} = (e_{13}, e_{23}, 0), \qquad \boldsymbol{\omega} = (u_1, u_2, w)$$

$$\langle \mathbf{a}, \mathbf{b}, \ldots, \mathbf{h} \rangle = \sum_{i=1}^{3} a_i b_i \ldots h_i$$

$$\gamma_i^+ \setminus \gamma_k^- = \gamma_{i,k}^{+,-}, \qquad d\gamma_{i,3}^{+,-} = (d\gamma_{1,3}^{+,-}, d\gamma_{2,3}^{+,-}, d\gamma_{3,3}^{+,-})$$

$$\mathbf{P}^\circ = (T^\circ, S^\circ, N^\circ), \qquad \mathbf{P}_1^\circ = (T_1^\circ, S_1^\circ, N_1^\circ)$$

$$\mathbf{Z} = (X, Y, Z), \qquad \mathbf{Z}_1 = (0, 0, Z_1)$$

and componentwise vector multiplication,

$$\mathbf{c} = \mathbf{ab}, \quad c_i = a_i b_i \qquad (i = 1, \ldots, l;\ l = 3, 6)$$

It will be convenient for us to deviate from the symbols used to denote strains and forces in [3],

$$\varepsilon_1 = \varepsilon_{11}, \quad \varepsilon_2 = 2\varepsilon_{12}, \quad \varepsilon_3 = \varepsilon_{22}, \quad \varepsilon_4 = \varkappa_{11}, \quad \varepsilon_5 = \varkappa_{12}, \quad \varepsilon_6 = \varkappa_{22}$$

$$S_1 = T_{11}, \quad S_2 = T_{12}, \quad S_3 = T_{22}, \quad S_4 = M_{11}, \quad S_5 = 2M_{12}, \quad S_6 = M_{22}$$

The general elasticity relations for anisotropic laminar shells with allowance for thermal expansion are

$$\mathrm{S} = B\boldsymbol{\varepsilon} - \mathbf{t}, \quad \mathbf{t} = (\mathbf{b}_1^{(\lambda)} + \mathbf{b}_3^{(\lambda)})\, T_0 + (\mathbf{b}_4^{(\lambda)} + \mathbf{b}_6^{(\lambda)})\, T_1 \qquad (1.3)$$

Here B is the positive-definite rigidity matrix whose coefficients are expressible in terms of the layer characteristics by means of formulas (8.3)–(8.5) of [6]; $\mathbf{b}_i^{(\lambda)}$ ($i = 1, 3, 4, 6$) is the ith column of the matrix $B^{(\lambda)}$ whose coefficients can be determined from the same formulas as in the case of the matrix B by replacing $B_{jk}{}^s$ in formulas (8.3)–(8.5) of [6] by $\lambda_s B_{jk}{}^s$ (λ_s is the coefficient of thermal expansion of the sth layer); T_0 is the temperature of the coordinate surface; T_1 is the temperature gradient.

It is convenient to set

$$\varepsilon_i = e_i + \theta_i = e_i + \sum_{j=1}^{2} \psi_{ij}\tau_{ij} \qquad (i = 1, \ldots, 6) \qquad (1.4)$$

where $e_i, \psi_{ij}, \tau_{ij}$ are the corresponding linear operators acting on ω. Formulas of this type are suitable for describing extremely diverse variants of deformation relations.

The following force facts are given:

$$Z,\ Z_1,\ \mathrm{k}\ \text{on}\ \sigma,\quad T^\circ,\ T_1^\circ,\ \beta_1,\ \text{on}\ \overset{+}{\gamma_1},$$
$$S^\circ,\ S_1^\circ,\ \beta_2\ \text{on}\ \overset{+}{\gamma_2},\ N^\circ,\ N_1^\circ,\ \beta_3\ \text{on}\ \overset{+}{\gamma_3},\ M^\circ,\ \beta_4\ \text{on}\ \overset{+}{\gamma_4}$$

Here $\mathbf{Z}_1$, $\mathbf{P}_1^\circ$ is the follower loading, $\mathbf{k}$ and β are the coefficients of elasticity of the supports.

The geometric boundary conditions are of the form

$$\langle \omega, \mathbf{n}_i\rangle|_{\gamma_i^-} = f_i \quad (i = 1, 2, 3) \qquad \langle \xi, \mathbf{n}_1\rangle|_{\gamma_4^-} = f_4 \tag{1.5}$$

The shell may be part of a structure, so that in addition to the geometric and static boundary conditions we must also consider the matching conditions. The ith geometric and the ith static matching conditions are specified on the set γ_i $(i = 1, 2, 3, 4)$. The plane measure of the sets γ_i, γ_i . $\overset{+}{\gamma_i}$ $(i = 1, 2, 3, 4)$ is equal to zero. Without limiting generality we can assume that

$$\gamma_i \cup \gamma_i^- \cup \gamma_i^+ = \Gamma \qquad (i = 1, 2, 3, 4)$$

The equations of equilibrium of the shell in terms of forces and moments appear in various papers and monographs [3] and need not be written out here, especially since they are derivable from the the virtual work principle formulated in Sect. 4.

2. Ancillary assumptions. In addition to the usual spaces $C(G)$, $L_p(G)$, $p \geqslant 1$, we shall also use [7] the Sobolev-Slobodetskii spaces $W_p^{(r)}(G)$ and $W_p^{(r-1/p)}(G)$ (r is an integer), and $V = W_2^{(1)}(G) \times W_2^{(1)}(G) \times W_2^{(2)}(G)$. Moreover, we set

$$\|f\|_{C(G)} = |f|, \quad \|f\|_{0,p,\Omega}^p = \int |f|^p\, d\Omega; \qquad \|f\|_{W_p^{(r)}(G)} = \|f\|_{r,p}$$

Norms in the spaces W and V are defined in a natural way.

Lemma 2.1. Let the functions occurring in Eqs. (1.5) satisfy the condition

$$f_i \in W_2^{(l_i - 1/p)}(\gamma_i^-), \qquad l_1 = l_2 = l_4 = 1,\ l_3 = 2 \qquad (i = 1, \ldots, 4)$$

In this case there exists a vector function $\omega^- \in V$ which satisfies (1.5) and vanishes near γ_i (the ith geometric matching condition).

The proof follows from continuation theorems [7] in self-evident fashion.

Let E be the closure of all vector functions ω smooth in $\bar{G}$ which satisfy homogeneous geometric boundary conditions (1.5), i.e. let it be the closure in norm of the space V. Let us introduce the bilinear form

$$A(\omega^{(1)}, \omega^{(2)}) = \int (\langle B\mathbf{e}^{(1)}, \mathbf{e}^{(2)}\rangle + b\,\langle \xi^{(1)}, \xi^{(2)}\rangle)\, dG_A \quad (b > 0,\ dG_A = A_1 A_2\, d\alpha_1 d\alpha_2) \tag{2.1}$$

It is clear that if $A(\omega, \omega) = 0$, then $\mathbf{e} = 0$ and $\xi = 0$. This means that ω is the displacement of the shell as a rigid whole.

Let $M \subset E$ be a subspace, $M = \{\omega : A(\omega, \omega) = 0\}$. Let us introduce the factor space $E^* = E / M$. By definition,

$$\|\omega\|_{E^*} = \inf \|\omega'\|_E \qquad (\omega' \in E,\ \omega \in E^*,\ \omega' \in \omega)$$

There exists a unique "normal" representative ω^* of the class of ω such that

$$\|\omega\|_{E^*} = \|\omega^*\|_E,\ \omega^* \in \omega$$

The space E is Hilbertian, which means that E^* is also Hilbertian.

Lemma 2.2, (see [2]). There exists a constant m such that

$$m\Omega \leqslant \|\omega\|_{E^*} \qquad \forall\omega\in E^* \qquad (m>0)$$

$$\Omega = |w^*|,\ \|f\|_{0,p,a}, \qquad f = \overset{*}{w}_{,1},\ \overset{*}{w}_{,2},\ u_1^*,\ u_2^*, \qquad a = G,\ \gamma \tag{2.2}$$

Here γ is a piecewise-smooth contour from G, and $1 \leqslant p < \infty$. Moreover, the relation expressed by inequality (2.2) is completely continuous, i.e. the boundedness of the set $\{\omega\}$ in E^* implies compactness in the sense of the left sides of (2.2).

Lemma 2.3 (in terms of [8]). For the elliptic problem

$$L(x, \partial/\partial x)\, u(x) = f(x) \quad (x\in\Omega), \qquad B(x, \partial/\partial x)\, u(x)|_S = \Phi(x')$$

we have the prior inequality

$$\sum_{j=1}^{m} \|u_j\|^{\Omega}_{p,\,l+t_j} \leqslant C\Big(\sum_{\substack{i=1\\ l\geqslant s_i}}^{m} \|f_i\|^{\Omega}_{p,\,l-s_i} + \sum_{\substack{q=1\\ l>\sigma_q}}^{r} \|\Phi_q\|^{S}_{p,\,l-\sigma_q-1/p}\Big) \tag{2.3}$$

Prior estimate (2.3) for the case $l-s_i \geqslant 0,\ l-\sigma_q > 0\ (i=1,\ldots,m;\ q=1,\ldots,r)$ appears in [8]; inequality (2.3) was obtained on the basis of the relevant results of [8].

Bilinear form (2.1) defines the scalar product and the norm in E^*,

$$(\omega^{(1)}, \omega^{(2)})_H = A(\omega^{(1)}, \omega^{(2)}), \qquad \|\omega\|_H^2 = \int (\langle B\mathrm{e}, \mathrm{e}\rangle + b\langle \xi, \xi\rangle)\, dG_A$$

Lemma 2.4. There exist constants m, m_1 such that

$$m\|\omega\|_{E^*} \leqslant \|\omega\|_H \leqslant m_1\|\omega\|_{E^*} \qquad (m, m_1 > 0) \tag{2.4}$$

uniformly for all $\omega\in E^*$.

The existence of m_1 is proved in [2]. It remains for us to show that

$$m = \inf_\omega (\|\omega\|_H \|\omega\|_{E^*}^{-1}) > 0, \qquad \omega\in E^*$$

Let $m=0$. In this case there exists a sequence $\{\omega^{(n)}\}$ such that

$$\|\omega^{(n)}\|_{E^*} = 1, \qquad \|\omega^{(n)}\|_H \to 0, \qquad \omega^{(n)} \to \omega^{(0)} \text{ weakly in } E^*$$

Since $\omega^{(0)}$ cannot be equal to zero [2], the existence of such a sequence together with (1.1) and Lemma (2.2) imply that

$$\|a^{(n)}\|_{0,2,G} \to 0, \qquad a = e_i \qquad (i=1,\ldots,6) \tag{2.5}$$

$$|w^{*(n)}|,\ \|a^{(n)}\|_{0,p,b} \to 0, \qquad a = u_1^*,\ u_2^*,\ \overset{*}{w}_{,1},\ \overset{*}{w}_{,2}, \qquad b = G,\ \gamma \tag{2.6}$$

$$\|g_i^{(n)}\|_{0,2,G} \to 0 \qquad (i=1,2) \tag{2.7}$$

Here g_1, g_2 correspond to the elliptic problem

$$\overset{*}{u}_{1,1} - \overset{*}{u}_{2,2} = g_1, \qquad A_2^{-1}\overset{*}{u}_{1,2} + A_1^{-1}\overset{*}{u}_{2,1} = g_2, \qquad \frac{\partial u_1^*}{\partial n_1}\Big|_\Gamma = 0$$

Expression (2.7) and Lemma (2.3) imply Item (a); Item (a), (2.5), (2.6), and (1.1) imply Item (b) and Item (c) of the following statement:

$$\text{a) } \|u_i^{*(n)}\|_{1,2} \to 0 \quad (i=1,2), \quad \text{b) } \|w^{*(n)}\|_{2,2} \to 0, \quad \text{c) } \|\omega^{(n)}\|_{E^*} \to 0$$

Item (c) contradicts the condition $\|\omega^{(n)}\|_{E^*} = 1$.

We denote the space E^* with the norm $\|\cdot\|_H$ by H.

Lemma 2.5. Lemma 2.2 remains valid if E^* is replaced by H throughout its formulation.

Let us introduce the sets $\gamma_i^\circ \subset \gamma_i^+$ and $G_i^\circ \subset G$ on which $\beta_i < 0$ and $k_i < 0$, respectively, hold almost everywhere ($i = 1, 2, 3$). By Lemma 2.4 there exist constants c_i^* ($i = 1, \ldots, 6$) such that

$$\| u_j^*, A \|^2_{0, q_1, G_j^\circ} \leqslant c_j^* \| \omega \|^2_H \quad (j = 1, 2) \quad (\| f, A \|^p_{0, p, G} = \int | f |^p \, dG_A)$$

$$\operatorname{mes} G_3^\circ \, | A_1 A_2 | \, | w^* |^2 \leqslant c_3^* \| \omega \|^2_H, \quad \| \langle \omega, \mathbf{n}_j \rangle \|^2_{0, q_1, \gamma_j^\circ} \leqslant c_{3+j}^* \| \omega \|^2_H$$

$$| w^* |^2 \leqslant c_6^* \| \omega \|^2_H, \qquad q_1 = 2q / (q - 1) \qquad q > 1$$

The discussion to follow rests on the following assumptions:

1) the linear operators ψ_{ij}, τ_{ij} (1.4) are completely continuous in H;
2) $\theta = 0$ if and only if $\xi = 0$;
3) if $\xi = 0$, then $\psi_{ij} = \tau_{ij} = 0$ ($i = 1, \ldots, 6$; $j = 1, 2$);
4) $B \in L_\infty (G)$, λT_0, $\lambda T_1 \in L_2 (G)$, X, Y, Z_1, k_1, $k_2 \in L_q (G)$

$$\langle \mathbf{P}^\circ, \mathbf{n}_i \rangle, \ \langle \mathbf{P}_1^\circ, \mathbf{n}_i \rangle, \ \beta_i \in L_q (\gamma_i^+) \ (i = 1, 2), \ M^\circ, \ \beta_4 \in L_q (\gamma_4^+)$$
$$k_3, \ Z \in C^* (G), \ N_1^\circ \in C^* (\gamma_3^+), \ L_q (\gamma^{+, -}_{3, 1}), \ L_q (\gamma^{+, -}_{3, 2})$$
$$N^\circ, \ \beta_3 \in C^* (\gamma_3^+) \qquad (q > 1)$$

5) there exist constants $c_i > c_i^*$ ($i = 1, \ldots, 6$) such that

$$c_1 \| k_1, A \|_{0, q, G_1^\circ} + c_2 \| k_2, A \|_{0, q, G_2^\circ} + c_3 \| k_3 \|_{C^* (G_3^\circ)} +$$
$$+ c_4 \| \beta_1 \|_{0, q, \gamma_1^\circ} + c_5 \| \beta_2 \|_{0, q, \gamma_2^\circ} + c_6 \| \beta_3 \|_{C^*, (\gamma_3^\circ)} < 1/2 \tag{2.8}$$

6) the conditions of Lemma 2.1 are fulfilled.

References to these assumptions will be made in the form "(5) (2.8)". Assumptions (1) (2.8) – (3) (2.8) are fulfilled for most deformation relations. It is usually possible to prove Lemma 2.4 by means of Lemma 2.3.

If the shell is considered as part of a structure, then its contribution to the equation of the virtual work principle is expressed by the functional

$$\Lambda^a (\omega^- + \omega, \omega^\wedge) = (\omega, \omega^\wedge)_H + Q_1^a (\omega, \omega^\wedge) + Q_2^a (\omega, \omega^\wedge)$$

$$Q_1^a (\omega, \omega^\wedge) = \int [\langle B (\mathbf{e}_a^- + \varphi^{, -} + \theta^a), (\mathbf{e}^\wedge + \varphi^{\wedge, -} + \varphi^{, \wedge a}) +$$
$$+ \langle B\mathbf{e}, (\varphi^{-, \wedge} + \varphi^{, \wedge a}) \rangle - \mathbf{t} (\mathbf{e}_a + \varphi_a^{-, \wedge} + \varphi^{, \wedge a}) - b \langle \xi, \xi^\wedge \rangle] \, dG_A$$

$$Q_2^a (\omega, \omega^\wedge) = - \int \langle [\mathbf{Z}_a - \mathbf{k} (\omega_a^- + \omega) - Z_1 (\eta_a^- + \eta)], \omega^\wedge \rangle \, dG_A -$$
$$- \int \langle [P_a^\circ + P_{1a}^\circ - \beta (\omega_a^- + \omega)^\circ], \omega^{\wedge, \circ}, d\gamma_i^+ \rangle -$$
$$- \int \langle P_1^\circ \xi_a^- + \xi)^\circ, w d\gamma^{+, -}_{i, 3} \rangle + \int N_1 \langle (\xi_a^- + \xi)^\circ, \omega^{\wedge, \circ}, d\gamma^{+, -}_{3, i} \rangle -$$
$$- \int [M_a^\circ - \beta_4 \langle \xi_a^- + \xi, \mathbf{n}_1 \rangle] \, \xi_1^{\wedge, \circ} \, d\gamma_4^+ \qquad (a = , +) \tag{2.9}$$

$$\varphi_i^{\alpha, \beta} = \sum_{j=1}^{2} (\psi_{i, j}^\alpha \tau_{ij}^\beta + \psi_{ij}^\beta \tau_{ij}^\alpha), \qquad \alpha, \beta = \quad , -, \wedge$$
$$y^- = y (\omega^-), \qquad y^\wedge = y (\omega^\wedge)$$

Here $\omega^\wedge$ is the virtual displacement under conditions (1.5). We assume that the right sides of formulas (2.9) for Λ, Q_1, Q_2 can be evaluated from the normal representatives [2]; the asterisks have been omitted.

Lemma 2.6. Let assumptions (2.8) be fulfilled. In this case the following representation is valid for all $\omega, \omega^\wedge \in H$

$$a) \quad Q_1(\omega, \omega^\wedge) = -(K_1\omega, \omega^\wedge)_H$$

$$b) \quad Q_2(\omega, \omega^\wedge) = -(K_2\omega, \omega^\wedge)_H$$

$$c) \quad \Lambda(\omega^- + \omega, \omega^\wedge) = (\omega - K\omega, \omega^\wedge)_H, \qquad K = K_1 + K_2$$

Here K_1, K_2, K are completely continuous operators in H.

Similar statements are proved by the functional method in [2], so we need only consider the distinctive features of the proof in the present case.

Let us consider the equation

$$P(\omega) = \int \langle [B\varepsilon, (e + \varphi_{\cdot}^- + \theta)\rangle + \langle Be, (\varphi_{\cdot}^- + \theta)\rangle + {}^1\!/_2 \langle B(\varphi_{\cdot}^- + \theta)(\varphi_{\cdot}^- + \theta)\rangle - t(e + \varphi_{\cdot}^- + \theta) - {}^1\!/_2 b \langle \xi, \xi\rangle]\, dG_A$$

It is not difficult to prove the weak continuity of $P(\omega)$ in H and the fact that $Q_1(\omega, \omega^\wedge)$ is the Gâteaux differential of $P(\omega)$. The subsequent statements of Item (a) can be proved as in [2]. Item (b) can also be proved by the method of [2].

3. Proof of the basic inequality. Let us introduce the additional sumbols

$$\|a\|^2 = \sum_{i=1}^{6} \|a_i\|_A^2, \quad \|g\|_A = \|g, A\|_{0,2,G}, \quad a^+ = Ra, \quad a_+ = R_a^{-1}$$

$$\Phi(\omega) = \Lambda(\omega^- + \omega, \omega), \quad \Phi^+(R, \omega_+) = R^{-2}\Phi(\omega) = \Lambda^+(\omega^- + \omega_+, \omega_+) \quad (\|\omega\|_H = R)$$

Lemma 3.1. Let conditions (2.8) be fulfilled. In this case there exist constants μ, ρ such that

$$\Phi(\omega) \geqslant \mu R^2 \quad \forall \|\omega\|_H = R \geqslant \rho \quad (\mu, \rho > 0) \tag{4.1}$$

This can be proved by the method developed in [2]. The statements

$$\Phi^+(R, \omega) \geqslant \mu \qquad \forall \|\omega\|_H = 1 \tag{4.2}$$

are equivalent to (4.1).

It is clearly sufficient to prove that

$$\lim \Phi^+(R, \omega) > 0 \qquad \text{as} \quad R \to \infty \qquad \forall \|\omega\|_H = 1 \tag{4.3}$$

uniformly in ω.

This implies the existence of the constant μ and the validity of (4.2) for sufficiently large R. Otherwise there would exist a sequence $\{R_m, \omega^{(n)}\}$ such that

$$\Phi^+(R_m, \omega^{(n)}) \to c \leqslant 0, \quad R_m \to \infty \quad \text{as} \quad n, m \to \infty \qquad (\|\omega^{(n)}\|_H = 1) \tag{4.4}$$

This contradicts (4.3), which implies the sufficiency of (4.3).

Let us suppose that (4.3) does not hold. This means that (4.4) is valid. The functional $\Phi^+(R, \omega)$ is a fourth-degree polynomial in ω with the leading term

$$\Phi_4^+(R, \omega) = 2\int \langle B\theta^+, \theta^+\rangle\, dG_A$$

and if the sequence $\{\|\theta_{n,m}^+\|\}$ is not bounded, then $\Phi^+(R_m, \omega^{(n)}) \to +\infty$ and (4.4) does not hold. This implies that the sequence $\{\|\theta_{n,m}^+\|\}$ is bounded, since we assumed that (4.4) holds; hence, $\|\theta_n\| \to 0$ as $n \to \infty$, since $R_m \to \infty$.

Thus, the sequence

$$\lim \| \theta^+_{n,m} \| \quad \text{as} \quad m, n \to \infty$$

has a limit; the limit over n for every fixed m also exists and is equal to zero. Recalling the repeated limit theorem, we obtain

$$\lim \| \theta^+_{n,m} \| = \lim_m \lim_n \| \theta^+_{n,m} \|, \qquad n, m \to \infty$$

By virtue of this fact there exists an $n_0(\varepsilon)$ such that

$$\Phi^+ (R_m, \omega^{(n)}) = \| \omega^{(n)} \|^2_H + \int \langle \mathrm{k}, \omega^{(n)}, \omega^{(n)} \rangle \, dG_A + \int \langle \beta \omega^{\circ\,(n)}, \omega^{\circ\,(n)} d\gamma^{\cdot} \rangle + f_{nm}$$
$$(| f_{nm} | \langle \varepsilon, n, m \rangle n_0)$$

Appealing to (5) (2. 8), we obtain

$$\Phi^+ (R_m, \omega^{(n)}) \geqslant {}^1\!/_2 \| \omega^{(n)} \|^2_H - \varepsilon, \qquad n, m \geqslant n_0 \qquad (\| \omega^{(n)} \|_H = 1)$$

This inequality clearly contradicts hypothesis (4.4), which implies the validity of the lemma.

4. Solvability of the problem on the equilibrium modes of an arbitrary anisotropic laminar shell. A shell satisfying the conditions of Sects. 1 and 2 will be called "regular". We call a shell "piecewise-regular" if it can be decomposed into a finite number of regular parts. Let σ be the coordinate surface of the shell and let $\sigma^{(k)}$ $(k = 1, \ldots, N)$ be its regular part. Let us also decompose the load action on the shell σ and the boundary conditions into parts corresponding to $\sigma^{(k)}$.

We can carry out the construction as follows. We supply all the quantities occurring in Sects. 1 – 3 with an additional index k. This will mean that the quantity so marked corresponds to the kth piece. The statements of Sects. 1 – 3 hold for the quantities accompanied by the index k. This means that

$$\sigma = \bigcup_{k=1}^{N} \sigma^{(k)}, \qquad G = \bigcup_{k=1}^{N} G^{(k)}, \qquad G^{(k)} = \{\alpha_1^{(k)}, \alpha_2^{(k)}\}$$

$$\Gamma = (\bigcup_{k=1}^{N} \Gamma^{(k)}) \setminus (\bigcup_{k=1}^{N} \bigcap_{i=1}^{4} \gamma_i^{(k)}), \qquad \gamma_i = 0 \qquad (i = 1, \ldots, 4)$$

Let the expression

$$\omega = (\omega^{(1)}, \ldots, \omega^{(N)})$$

mean that

$$\omega = \omega^{(k)} \text{ on } G^{(k)} \qquad (k = 1, \ldots, N)$$

and that the geometric matching conditions in terms of the quantities $\omega^{(n)}$ and $\omega^{(m)}$ $(n, m = 1, \ldots, N)$ are fulfilled on $\gamma_i^{(n)} \cap \gamma_i^{(m)}$ $(i = 1, \ldots, 4)$. Let us introduce the direct product of spaces

$$H = \prod_{k=1}^{N} \dot{\times} H^{(k)} \qquad (M^{(k)} = M, \, k = 1, \ldots, N)$$

where $H^{(k)}$ is the space defined in Sect. 2.

The equilibrium condition for the shell can be expressed with the aid of the virtual work principle by means of the equation

$$\Lambda (\omega^{-} + \omega, \omega^{\wedge}) = \sum_{k=1}^{N} \Lambda^{(k)} (\omega^{-(k)} + \omega^{(k)}, \omega^{\wedge\,(k)}) = 0 \qquad (5.1)$$

Definition 4.1. The "generalized solution" of the problem for a piecewise-regular shell is a function $\omega^{-} + \omega$ $(\omega \in H)$ which satisfies Eq. (5.1) for all $\omega^{\wedge} \in H$.

Lemma 4.1. Equation (5.1) is equivalent to the following operator equation in H:

$$\Lambda(\omega^{-} + \omega, \omega^{\wedge}) = (\omega - K\omega, \omega^{\wedge}) = 0 \quad \text{or} \quad \omega - K\omega = 0$$

where the completely continuous operator

$$K = \sum_{k=1}^{N} \dotplus K^{(k)}$$

is the direct sum of the operators $K^{(k)}$ acting in $H^{(k)}$. The proof follows from Lemma 2.6.

Lemma 4.2. For a piecewise-regular shell there exist positive constants R_0, μ such that

$$\Phi(\omega) = \sum_{k=1}^{N} \Phi^{(k)}(\omega^{(k)}) \geqslant \mu R^2, \qquad \forall \, \|\omega\|_H = R \geqslant R_0 \tag{5.2}$$

By virtue of Lemma 3.1 there exist a μ_k and a ρ such that

$$R_k^{-2}\Phi^{(k)}(\omega^{(k)}) \geqslant \mu_k \qquad \forall \, \|\omega^{(k)}\|_{H^{(k)}} = R_k \geqslant \rho \qquad (k = 1, \ldots, N)$$

It is clear that

$$R^{-2}\Phi(\omega) = \sum_{k=1}^{N}{}' (R_k / R)^2 [\Phi^{(k)}(\omega^{(k)}) / R_k^2] \geqslant \sum_{R_k \geqslant \rho} \mu_k (R_k / R)^2 +$$

$$+ \sum_{R_k < \rho} (\mu_k - 1)(\rho / R)^2 \qquad \Big(\sum_{R_k \geqslant \rho} R_k^2 \geqslant R^2 - N\rho\Big) \tag{5.3}$$

Expression (5.3) implies that

$$R^{-2}\Phi(\omega) \geqslant \inf \mu_k - N(\rho / R)^2 \sum_{k=1}^{N} \mu_k + (\rho / R)^2 \sum_{R_k < \rho} (\mu_k - 1)$$

Hence, there exists an $R_0 > 0$ such that

$$R^{-2}\Phi(\omega) \geqslant \inf_k \mu_k - \varepsilon = \mu, \qquad (\rho / R)^2 \left| \left[N \sum_{k=1}^{N} \mu_k + \sum_{R_k < \rho} (\mu_k - 1) \right] \right| < \varepsilon, \qquad R > R_0$$

The statement has been proved.

The following statements follow from inequality (5.2) exactly as in [2].

Lemma 4.3. The rotation of the completely continuous field $\omega - K\omega$ on a sphere $S(R, 0)$ of sufficiently large radius R is equal to +1.

Theorem 4.1. If the shell is piecewise-regular, then there exists a generalized solution of the problem in the sense of Definition 4.1 and $\|\omega\|_H < R_0$.

The above results enable us to investigate the differential properties of the solution.

By virtue of the theorems of [3], Lemma 4.3 guarantees the convergence of the Galerkin method and other projective methods.

The existence theorem of [2] for a closed cylindrical shell is readily obtainable from Theorem 4.1. In this case it is sufficient to cut the shell into two regular pieces by a diametric plane and to formulate the matching conditions at the cuts.

Theorem 4.1 remains valid if a different (linear, nonlinear) calculation theory is applied over each piece.

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EQUATIONS OF PERTURBED MOTION OF A BODY WITH A THIN-WALLED ELASTIC SHELL PARTIALLY FILLED WITH A LIQUID

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Linear equations of perturbed motion of a thin-walled elastic shell partially filled with a heavy compressible fluid considered in the acoustic approximation are derived; the principal [force] vector and the principal moment of the reactions exerted by the shell on the "carrying body" are determined. Perturbed motion with small vibrations is characterized by the displacement of a certain point attached to the rigid shell fastening contour, by rotation relative to this point, and by elastic displacements expressed as an expansion in the proper vibration modes of the fastened fluid-containing shell. The natural frequencies and vibration modes of a fluid-containing shell are determined by means of a variational principle.

Allowance for the compressibility of the fluid makes it possible to consider vibrations in the acoustic frequency spectrum. Moreover, calculations show that it may be necessary to make allowance for it in calculating the lower frequencies of the elastic vibrations of the shell, e.g. of the axisymmetric vibrations of relatively thick shells of revolution. Allowance for gravity is necessary in considering vibrations in the frequency spectrum of gravitational surface waves and vibrations of flexible fluid-containing shells.